B.Sc. (Maths) – part III
paper VI
Topic :- centre of a group (Group theory)

Central element in a group

Definition :- The set $Z(G)$ of all elements x of G such that $xg = gx$ for all $g \in G$ is called the centre of G. Symbolically
$$Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\}$$

Theorem :- The centre $Z(G)$ of a group G is a normal subgroup of G.

proof – we have $Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\}$

First we shall prove that $Z(G)$ is a subgroup of G.
For this we shall to show that
$x_1, x_2 \in Z(G) \Rightarrow x_1 x_2^{-1} \in Z(G)$
So let $x_1, x_2 \in Z(G)$ then
$x_1 g = g x_1$ and $x_2 g = g x_2$ for all $g \in G$.
we have $x_2 g = g x_2$ for all $g \in G$
$$\Rightarrow x_2^{-1}\{x_2 g\}x_2^{-1} = x_2^{-1}\{g x_2\}x_2^{-1}$$
$$\Rightarrow (x_2^{-1}x_2) g x_2^{-1} = x_2^{-1} g (x_2 x_2^{-1})$$

$\Rightarrow g x_2^{-1} = x_2^{-1} g$ for all $g \in G$

$\Rightarrow x_2^{-1} Z(G)$

Now $(x_1 x_2^{-1}) g = x_1 (x_2^{-1} g) = x_1 (g x_2^{-1})$, by definition

$= (x_1 g) x_2^{-1} = (g x_1) x_2^{-1} = g(x_1 x_2^{-1})$

$\therefore x_1 x_2^{-1} \in Z(G)$

Thus $x_1, x_2 \in Z(G) \Rightarrow x_1 x_2^{-1} \in Z(G)$

Hence $Z(G)$ is subgroup of G.

Now we shall have to prove that $Z(G)$ is a normal subgroup of G.

Let $x \in Z(G)$ and $g \in G$ we shall need to prove that $g x g^{-1} \in Z(G)$

we have $g x g^{-1} = (gx) g^{-1}$

$= (xg) g^{-1} \quad \because gx = xg$

$= x \in Z$

Thus $g \in G$, $x \in Z(G) \Rightarrow g x g^{-1} \in Z(G)$

$Z(G)$ is normal subgroup of G.

__Theorem__:- If $Z(G)$ be the centre of group G and $G/Z(G)$ be cyclic prove that G must be abelian.

Solution:- It is given that G/Z is cyclic. Let Zg be a generator of the cyclic group G/Z where g is some element of G

Let $a, b \in G$ then to prove that $ab = ba$. Since $a \in G$ therefore $Za \in G/Z$ But G/Z is cyclic having Zg as generator. Therefore there exists some integer m such that $Za = (Zg)^m = Zg^m$ because Z is normal subgroup of G.

Now $a \in Za$ therefore $Za = Zg^m$

$\Rightarrow a \in Zg^m \Rightarrow a = z_1 g^m$ for some $z_1 \in Z$

Similarly $b = z_2 g^n$ where $z_2 \in Z$ and n is some integer

Now $ab = (z_1 g^m)(z_2 g^n) = z_1 g^m z_2 g^n$

$= z_1 z_2 g^m g^n$ $(\because z_2 \in Z \Rightarrow z_2 g^m = g^m z_2)$

$= z_1 z_2 g^{m+n}$

Again $ba = z_2 g^n z_1 g^m = z_1 z_2 g^n g^m$

$= z_2 z_1 g^{m+n}$

$[\because z_1 \in Z \Rightarrow z_1 z_2 = z_2 z_1]$

$\therefore ba = z_1 z_2 g^{m+n}$

$\therefore ab = ba$

Since $ab = ba \ \forall \ a, b \in G$

Therefore G is abelian

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